Introduction into Quantum Theory of Magnetism

- ❑ Classification of magnetic materials
- ❑ Bohr-Van Leeuwen theorem (absence of magnetism in classical physics)
- ❑ Hydrogen molecule. Exchange interaction. Dirac's vector model
- ❑ Four main models for strong magnetism:
  - Ising model;
  - Heisenberg model;
  - Hubbard model;
  - s-d – model.

❑ Classification of magnetic materials

Any system may be characterized by its response to external stimuli. We shall be concerned mainly with the response of substances to a magnetic field ***H***. In this case the "output" is the magnetization ***M*** and the response function is the magnetic susceptibility χ.

$$M = \chi H, \chi = \chi(H,T)$$

• Diamagnetic substances: $\chi < 0, \chi \sim 10^{-5}$ molar susceptibility at room temperature

• Paramagnetic substances: $\chi > 0, \chi = C/T,$ (Curie law);

$\chi \sim 10^{-2}$ molar susceptibility at room temperature

❑ Classification of magnetic materials

•Strong magnetism: ferromagnetism, antiferromagnetism, ferrimagnetism.

$$\chi \to \infty$$

•Spontaneous magnetization, ***phase transition*** at critical temperature.
T_C, T_N are Curie and Neel temperatures for ferromagnets and antiferromagnets (ferrimagnets) respectively.

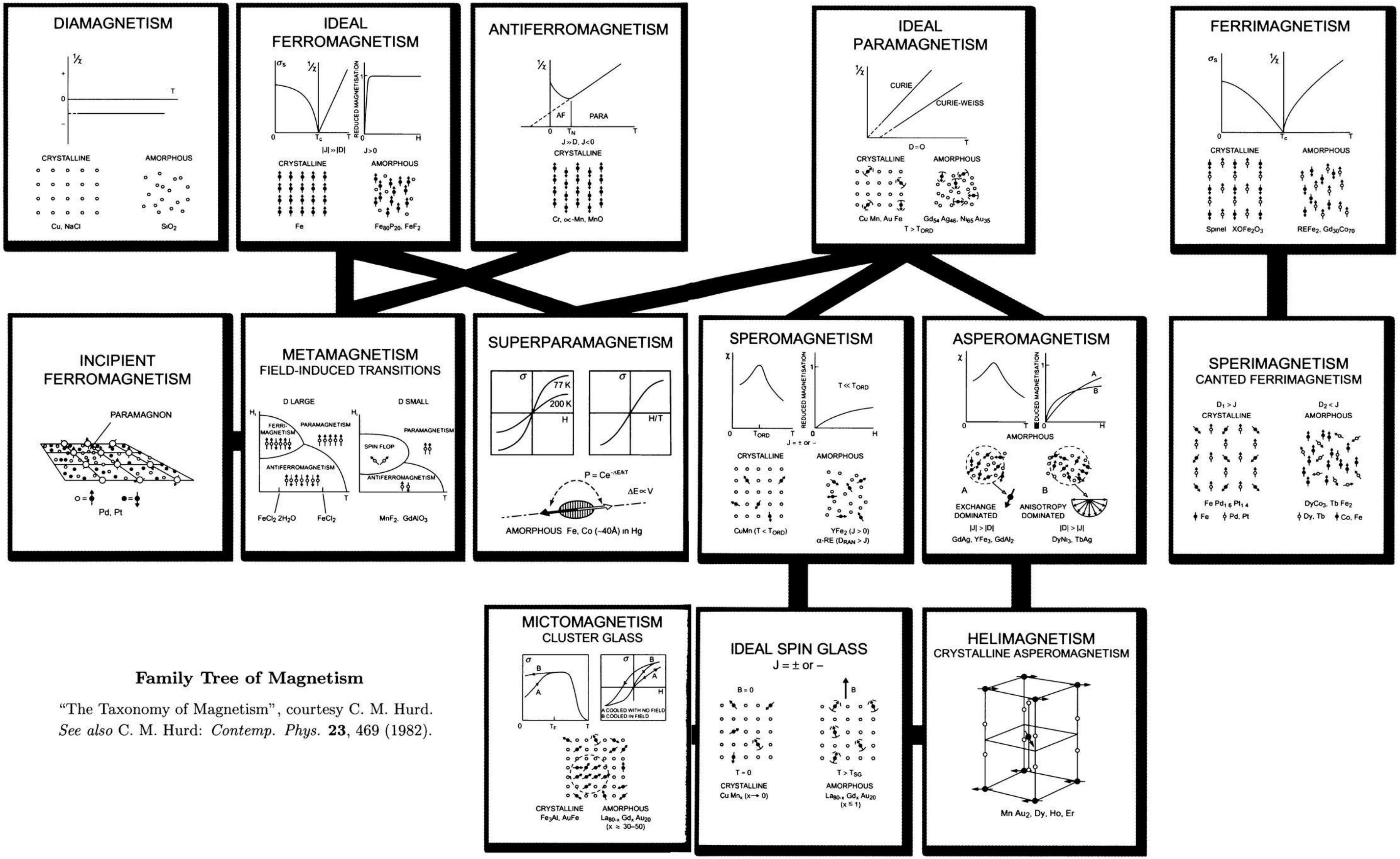
•Paramagnetic behavior at high temperatures above critical temperature (Curie-Weiss law)

$$\chi = C/\left(T - T_c\right)$$

$$\chi = C/\left(T + \theta_c\right)$$

for ferro- and antiferromagnetic materials respectively.

Table of Magnets



Family Tree of Magnetism

"The Taxonomy of Magnetism", courtesy C. M. Hurd.
See also C. M. Hurd: *Contemp. Phys.* **23**, 469 (1982).

❑ Bohr -Van Leeven theorem (absence of magnetism in classical physics):

At any finite temperature, and in all finite applied electrical or magnetic fields, the net magnetization of a collection of electrons in thermal equilibrium vanishes identically.

Bohr (1911), Van Leeuwen (1919)

Quantization of energy levels (quantum mechanics) is necessary for magnetism!

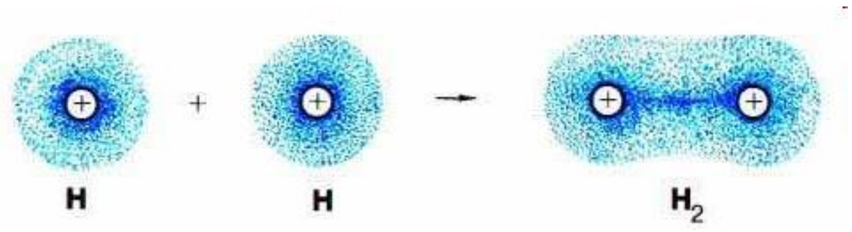
Which interactions are responsible for strong magnetism?

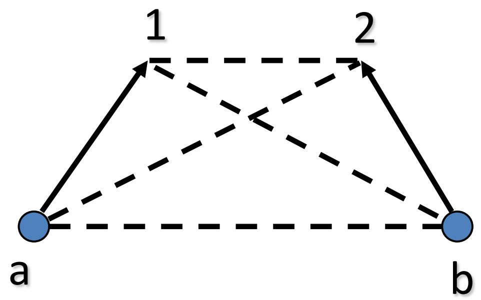
✘ Magnetic dipolar interactions $E_{d-d} \sim \dfrac{\mu_B^2}{a^3} \sim 10^{-16} erg$ are too weak

☑ Electrostatic interactions $E_e \sim \dfrac{e^2}{a} \sim 10^{-12} erg$ fit to typical values

$$kT_c \sim 10^{-12} erg;\ T_c \sim 1000K.$$

Hydrogen Molecule

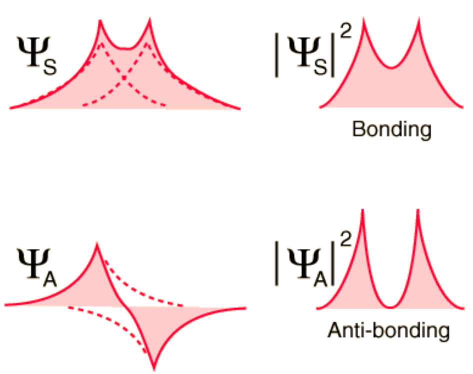




$$\hat{\mathbf{H}} = \frac{\mathbf{p}_1^2}{2m} - \frac{e^2}{r_{a1}} + \frac{\mathbf{p}_2^2}{2m} - \frac{e^2}{r_{b2}} + e^2\left(\frac{1}{r_{ab}} + \frac{1}{r_{12}} - \frac{1}{r_{a2}} - \frac{1}{r_{b1}}\right)$$

$\uparrow\downarrow, \quad S = 0, S_z = 0$ - «singlet»

$\uparrow\uparrow \quad S = 1, S_z = -1, 0, 1$ - «triplet»



Hydrogen Molecule

$\uparrow\downarrow$ "singlet" χ_a; $\psi_s = \dfrac{1}{\sqrt{2\left(1+l^2\right)}}\left(\varphi_{a1}\varphi_{b2} + \varphi_{a2}\varphi_{b1}\right)$;

$\uparrow\uparrow$ "triplet" χ_s; $\psi_a = \dfrac{1}{\sqrt{2\left(1-l^2\right)}}\left(\varphi_{a1}\varphi_{b2} - \varphi_{a2}\varphi_{b1}\right)$;

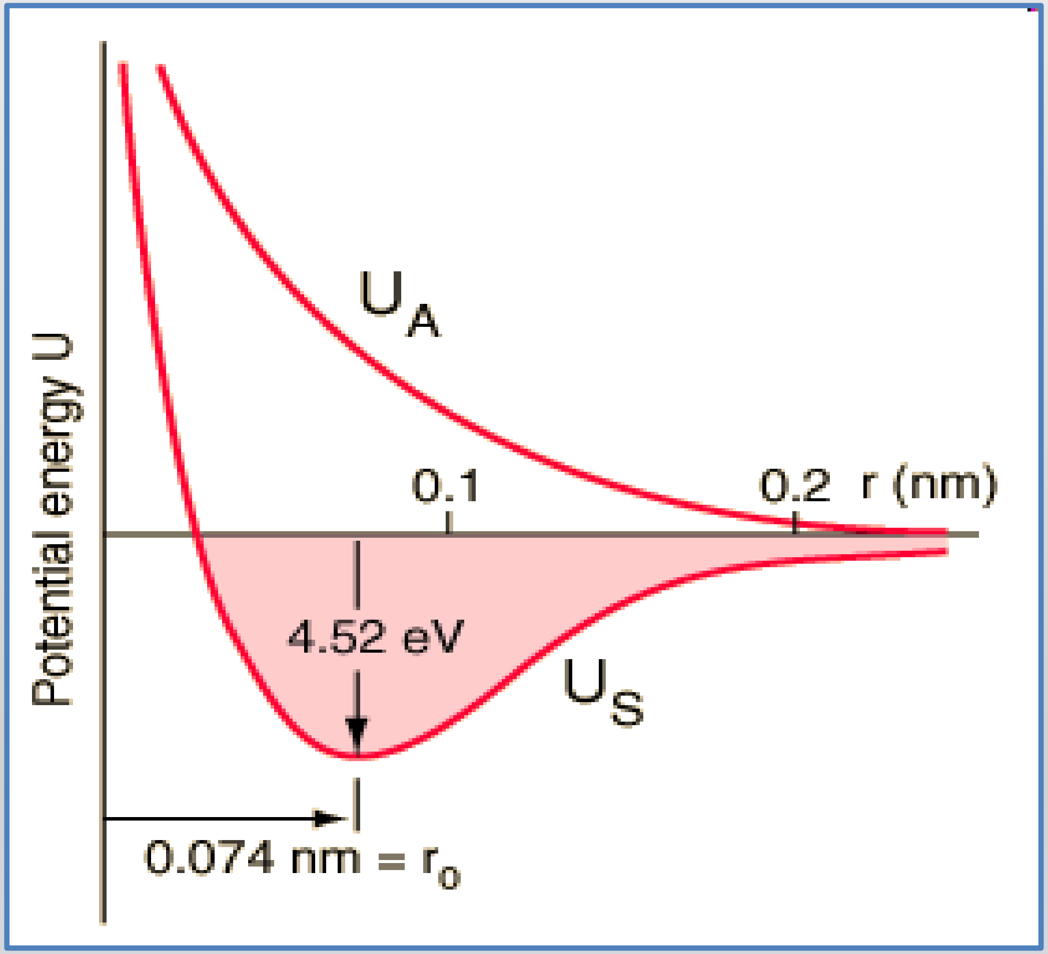
$l = \int \varphi_{a1}\varphi_{b1} dV_1$ is "overlap" integral

$A(r) = \int \varphi_{a1}\varphi_{b2} U \varphi_{a1}\varphi_{b2} dV_1 dV_2$ is electrostatic term

$B(r) = \int \varphi_{a1}\varphi_{b2} U \varphi_{a2}\varphi_{b1} dV_1 dV_2$ is "exchange" term

$$U = e^2\left(\frac{1}{r_{ab}} + \frac{1}{r_{12}} - \frac{1}{r_{a2}} - \frac{1}{r_{b1}}\right)$$

Hydrogen Molecule



$$E_{\uparrow\uparrow} = U_A = \frac{A(r) - B(r)}{1 - l^2};$$

$$E_{\uparrow\downarrow} = U_S = \frac{A(r) + B(r)}{1 + l^2};$$

HEITLER W., LONDON F. *Wechselwirkung neutraler Atome und homöopolare Bindung nach der Quantenmechanik* // Zeitschrift für Physik – 1927. – B. 44. – C. 455-472.

HEISENBERG W. *Zur Theorie des Ferromagnetismus.* // Zeitschrift für Physik – 1928.–V. 49.– p.619 – 636.

Dirac's vector model

Dirac in 1929 proposed spin Hamiltonian for H_2 molecule, equivalent to above perturbation theory consideration for microscopic Hamiltonian

$$\mathbf{H} = -J(r)\mathbf{S}_1\mathbf{S}_2 + E(r),\ \ \mathbf{S}_{1,2} = \left(S_{1,2}^x, S_{1,2}^y, S_{1,2}^z\right),\ s = \tfrac{1}{2};$$

$$\langle \mathbf{S}_1\mathbf{S}_2 \rangle = \frac{1}{2}\left\langle \left[(\mathbf{S}_1 + \mathbf{S}_2)^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 \right] \right\rangle = \frac{1}{2}S(S+1) - \frac{3}{4} = \begin{cases} -\dfrac{3}{4}, & S = 0; \\ \dfrac{1}{4}, & S = 1. \end{cases}$$

Exact eigen values of Dirac's exchange Hamiltonian are

$$E_{\uparrow\downarrow} = \frac{3}{4}J(r) + E(r); \quad E_{\uparrow\uparrow} = -\frac{1}{4}J(r) + E(r);$$

Here $J(r)$ is an exchange energy and $E(r)$ is an average electrostatic energy.
For H_2 $J(r) < 0$.

P. A. M. DIRAC, *Quantum Mechanics of Many-Electron Systems* . Proc. R. Soc. Lond., 1929. –V.123. – P.714-733.

Dirac's vector model

$$J(r) = E_{\uparrow\downarrow} - E_{\uparrow\uparrow} = 2\left(B(r) - l^2 A(r)\right) \Big/ \left(1 - l^4\right)$$

$$E(r) = \frac{1}{4} E_{\uparrow\downarrow} + \frac{3}{4} E_{\uparrow\uparrow}$$

According to modern ideas the strong magnetism of solids exists due to ***exchange interaction***s between electrons. It appears in those cases when the crystal lattice includes the atoms with partly filled inner shells. Such materials become spontaneously magnetized at rather low temperatures. This is one of the examples of the cooperative phenomena. ***We can not neglect this interaction at any reasonable approximation. So we have to solve many-body problem.***

Mainly, the progress in this field is connected with the constructions of a large number of lattice models. In these models, the magnetic interactions more of less simplified. There are four main models for strong magnetism, such as **Ising model**, **Heisenberg model**, **Hubbard model**, **s-d – model**.

From Dirac's vector model to Heisenberg exchange Hamiltonian

$$\mathbf{H}_{exchange} = -\sum_{i \neq j} J_{ij} \mathbf{S}_i \mathbf{S}_j , \quad \mathbf{S}_{i,j} = \left(S_{i,j}^x , S_{i,j}^y , S_{i,j}^z \right), \; s = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$J_{ij} = J\left(| \mathbf{r}_i - \mathbf{r}_j | \right)$ are exchange integrals for pairs of localized spins.

$J_{ij} > 0$ corresponds to ferromagnetic case and $J_{ij} < 0$ corresponds to antiferromagnetic one. J_{ij} decrease exponentially as a function of the atomic distances.

Heisenberg exchange Hamiltonian is adequate only for insulators because it takes into account only localized magnetic moments.

❑Four main models for strong magnetism

Heisenberg exchange Hamiltonian for magnetic dielectrics

$$\mathbf{H} = -g\mu_B H \underbrace{\sum_i S_i^z}_{\text{Zeeman energy}} - \underbrace{\sum_{i\neq j} J_{ij}\mathbf{S}_i \mathbf{S}_j}_{\text{Exchange energy}},$$

$$\mathbf{S}_i = \left(S_i^x, S_i^y, S_i^z\right),\; J_{ij} = J\left(|\vec{r}_i - \vec{r}_j|\right)$$

g is Landee factor, μ_B is Bohr magneton

$$\left[\mathbf{H}, \left(\sum \mathbf{S}_i\right)^2\right] = 0; \quad \left[\mathbf{H}, \sum S_i^z\right] = 0.$$

Total momentum and *z*-projection of total momentum are good quantum numbers

❑Four main models for strong magnetism

Low temperature properties of Heisenberg ferromagnet with nearest neighbor interactions ($J > 0$)

$$\mathbf{H} = -g\mu_B H \sum_{\vec{m}} S_{\vec{m}}^{z} - J \sum_{\vec{m},\vec{\delta}} \mathbf{S}_{\vec{m}} \mathbf{S}_{\vec{m}+\vec{\delta}},$$

Ferromagnetic ground state:

$$\mathbf{S}_{\vec{m}} \mathbf{S}_{\vec{m}+\vec{\delta}} = S_{\vec{m}}^{x} S_{\vec{m}+\vec{\delta}}^{x} + S_{\vec{m}}^{y} S_{\vec{m}+\vec{\delta}}^{y} + S_{\vec{m}}^{z} S_{\vec{m}+\vec{\delta}}^{z} =$$

$$\frac{1}{2}\left(S_{\vec{m}}^{-} S_{\vec{m}+\vec{\delta}}^{+} + S_{\vec{m}+\vec{\delta}}^{-} S_{\vec{m}}^{+}\right) + S_{\vec{m}}^{z} S_{\vec{m}+\vec{\delta}}^{z}$$

$$S_{\vec{m}}^{\pm} = S_{\vec{m}}^{x} \pm i S_{\vec{m}}^{y}; \quad S_{\vec{m}}^{+} \left|0\right\rangle = 0; \quad \mathbf{H}\left|0\right\rangle = E_{ground}\left|0\right\rangle;$$

$$E_{ground} = -g\mu_B sHN - Js^2 zN; \quad S_{total}^{z} = Ns.$$

❑Four main models for strong magnetism

The one magnon states for Heisenberg spin-s model $\left(S^z_{total} = Ns - 1\right)$

$$\mathbf{H}\left|1\right\rangle = \left(E_{ground} + \varepsilon\right)\left|1\right\rangle; \quad \left|1\right\rangle = \sum_{\vec{m}} A_{\vec{m}} S^-_{\vec{m}} \left|0\right\rangle;$$

$$\left(\varepsilon - g\mu H - 2Jsz\right) A_{\vec{m}} + 2Js \sum_{\vec{\delta}} A_{\vec{m}+\vec{\delta}} = 0;$$

$$\begin{cases} A_{\vec{m}} = N^{-1/2} \exp\left(i\vec{k}\vec{m}\right); \\ \varepsilon_{\vec{k}} = g\mu H + 2Js\left(z - \sum_{\vec{\delta}} \exp\left(i\vec{k}\vec{\delta}\right)\right). \end{cases}$$

$\vec{k}$ is the quasi-momentum of "magnon" (magnetic quasi-particle)

❑Four main models for strong magnetism

The one magnon states for Heisenberg spin-s model.
Long wave limit:

$$k << 1, \quad \varepsilon_{\vec{k}} \approx g\mu_B H + 2Jsk^2;$$

$$H = 0, \quad \varepsilon_{\vec{k}} \geq 0.$$

1D spin -1/2 chain with nearest neighbors interactions.
One-magnon states for periodic boundaries.

$$\left(\varepsilon - 2\mu H - 2J\right)A_m + J\left(A_{m+1} + A_{m-1}\right) = 0;$$

$$\begin{cases} A_m = N^{-1/2}\exp\left(ikm\right); \\ \varepsilon = 2\mu H - 2J\left(1 - \cos k\right), \quad k = \dfrac{2\pi l}{N}, l = 0, 1, N-1. \end{cases}$$

❑Four main models for strong magnetism

The two magnon spectrum for 1D spin-1/2 Heisenberg chain

$$\mathbf{H}\left|2\right\rangle = \left(E_{ground} + \varepsilon\right)\left|2\right\rangle; \quad \left|2\right\rangle = \sum_{m \neq n} A_{mn} S_m^- S_n^- \left|0\right\rangle;$$

$$A_{mn} = \exp\left[ik\left(m+n\right)/2\right] f_{m-n}; \quad A_{mn} = A_{nm};$$

$$\begin{cases} \left(\varepsilon - 4\mu H - 4J\right) f_l + 2J\cos\dfrac{k}{2}\left(f_{l+1} + f_{l-1}\right) = 0; \ \ l > 1; \\ \left(\varepsilon - 4\mu H - 2J\right) f_1 + 2J\cos\dfrac{k}{2} f_2 = 0; \end{cases}$$

$$f_l = Ax^l + Bx^{-l}; \quad \varepsilon = 4\mu H + 4J\left[1 - \cos\frac{k}{2}\left(x + \frac{1}{x}\right)\right];$$

❑Four main models for strong magnetism

The two magnon spectrum for 1D spin-1/2 infinite Heisenberg chain

Continuous spectrum.
Wave function obeys Pauli exclusion principle.

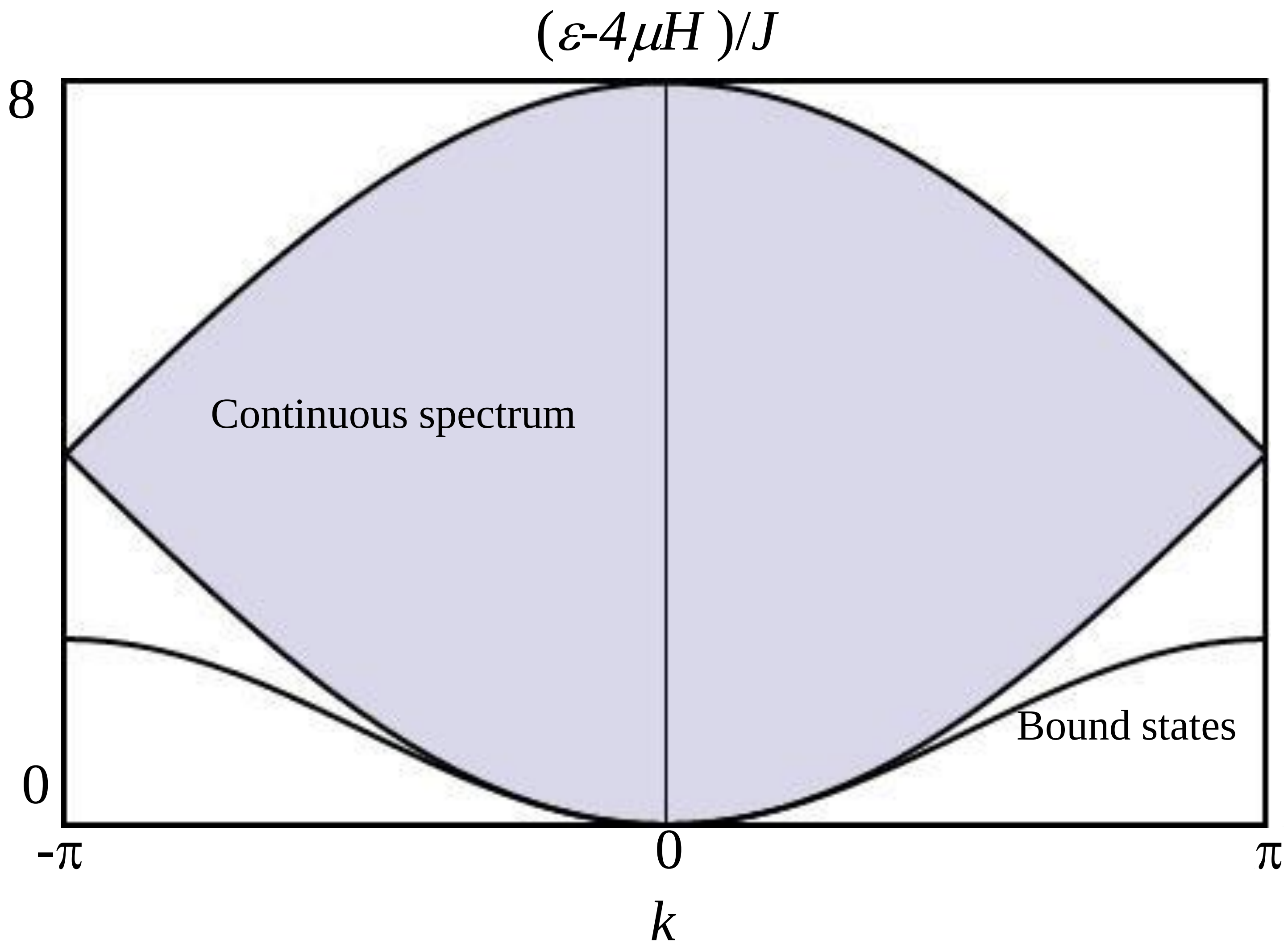
$$\varepsilon_{kq} = 4\mu H + 4J\left(1 - \cos\frac{k}{2}\cos q\right); \quad x = \exp(iq).$$

“Bound states”. Wave function decrease exponentially with the distance between inverted spins and does not obey Pauli exclusion principle.

$$\varepsilon_k = 4\mu H + J(1 - \cos k); \quad |x| < 1, \; x - real$$

❑Four main models for strong magnetism

Two magnon spectrum for spin-1/2 infinite 1D chain (***J* > 0**)



❑Four main models for strong magnetism

The two magnon spectrum for 1D spin-1/2 finite Heisenberg chain with periodic boundaries.

$$\varepsilon_{kx} = 4\mu H + 4J\left[1 - \frac{1}{2}\cos\frac{k}{2}\left(x + \frac{1}{x}\right)\right];$$

$$\underline{\underline{A_{mn} = A_{n+Nm};}} \quad \left(C_1 e^{ikN/2} - C_2\right)x^{n-m} + \left(C_2 e^{ikN/2} - C_1\right)x^{m-n};$$

$$C_2 = C_1 e^{ikN/2} = C_1 e^{-ikN/2}; \quad e^{ikN} = 1, \quad k = \frac{2\pi l}{N};$$

$$l = -\frac{N}{2},\ldots,0,\ldots,\frac{N}{2} - 1 \ \textit{for even } N,$$

$$l = -\frac{N-1}{2},\ldots,0,\ldots,\frac{N-1}{2} \ \textit{for odd } N.$$

❑Four main models for strong magnetism

The two magnon spectrum for 1D spin-1/2 finite Heisenberg chain with periodic boundaries.

$$f_l = C_1\left[x^l + (-1)^N x^{N-l}\right]; \quad l = m - n;$$

<u>Dispersion equation for *x*:</u>

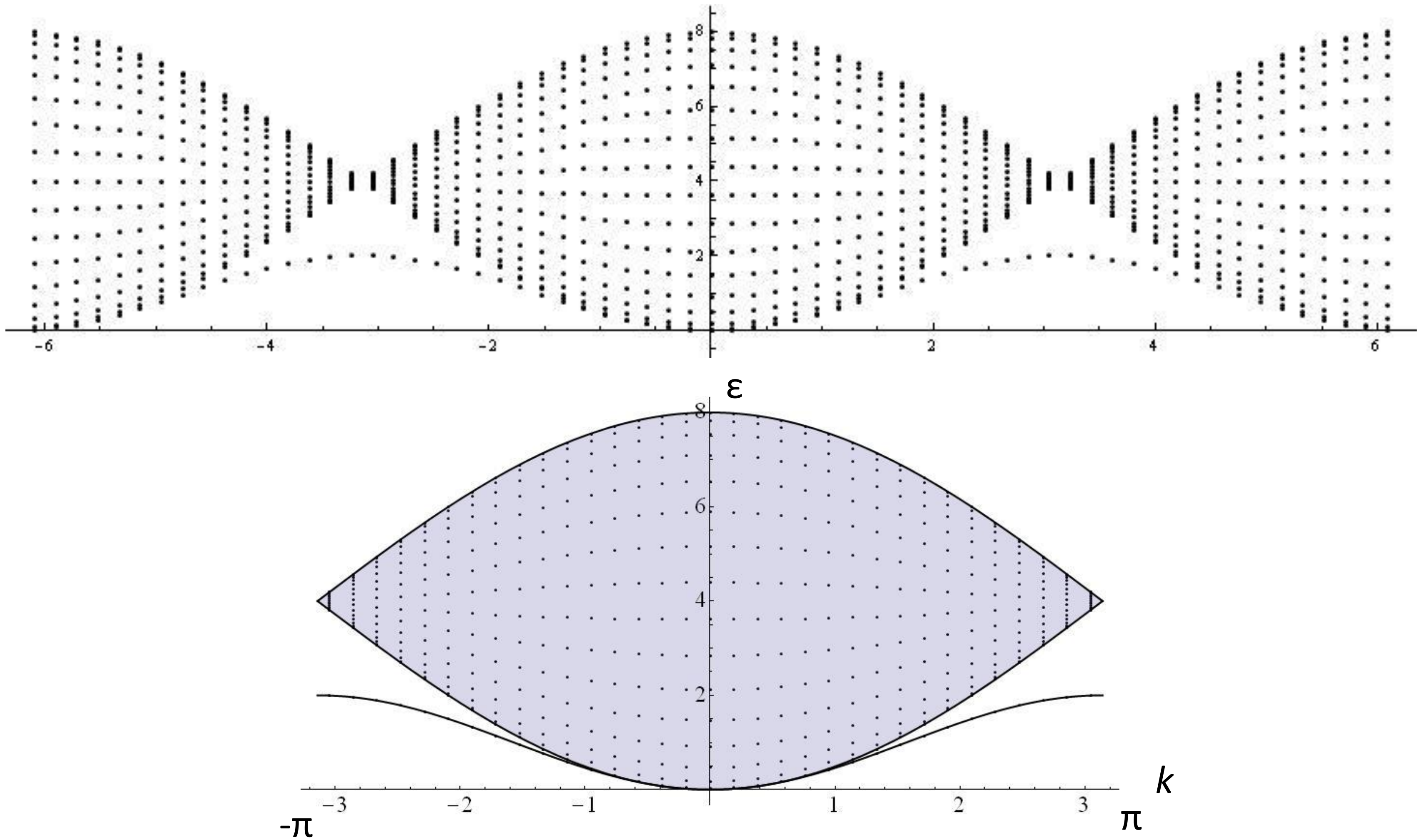
$$x\left[1+(-1)^N x^{N-2}\right] - \cos\left(\frac{\pi l}{N}\right)\left[1+(-1)^N x^N\right] = 0.$$

$$x = \exp(iq)$$

$1. \, |x| = 1,\ q$ is real;

$2. \, |x| < 1,\ x$ is real, so q is imaginary

The two magnon spectrum for 1D spin-1/2 finite Heisenberg chain with periodic boundaries.



❑Four main models for strong magnetism

Bethe wave function (WF) (ansatz Bethe, 1931)

For two magnons WF consists of 2! = 2 terms:

$$A_{n_1 n_2} = C\left(e^{ik_1 n_1 + ik_2 n_2 + \left(\frac{1}{2}\right) i\psi_{12}} + e^{ik_2 n_1 + ik_1 n_2 + \left(\frac{1}{2}\right) i\psi_{21}}\right),\ k_{1,2} = k/2 \mp q;$$

$$\underline{\varepsilon_{k_1,k_2} = \varepsilon_{k_1} + \varepsilon_{k_2};}$$

$$2ctg\left(\frac{\psi_{k_1 k_2}}{2}\right) = ctg\left(\frac{k_1}{2}\right) - ctg\left(\frac{k_2}{2}\right);$$

$$\varepsilon_{k_{1,2}} = 2\mu_B H - 2J\left(1 - \cos k_{1,2}\right).$$

$$k_{1,2} = \frac{2\pi p \pm \psi_{12}}{N}; \quad \psi_{12} = \psi_{21}.$$

Thus, a two magnon state is described by two wave vectors and a phase shift. There is refraction, but no scattering (back scattering).

Energy ε is just the energy of two noninteracting magnons. Interactions are included implicitly *via* the phase shifts.

❑Four main models for strong magnetism

Bethe wave function (WF) (ansatz Bethe, 1931)

Bethe's ansatz consists partly in the statement that in many-particle state, the wave functions subject only be phase shifts that are simply given, as the sum of two-particle shifts (there are only two-particle sollisions)

So, *r*-magnon WF consists of r! terms:

$$A_{n_1 n_2 \ldots n_r} = \sum P_{k_1, k_2, \ldots, k_r} \exp i\left(k_1 n_1 + k_2 n_2 \ldots + k_r n_r + 1/2 \sum_{j<}^{r} \sum_{t}^{r} \psi_{k_j, k_t} \right);$$

$A_{n_1 n_2 \ldots n_r}$ consists of a product over r plane-wave factors, summed over all permutations of k's with the ψ's antisymmetric over their subscripts and satisfying the equations

$$2ctg\left(\frac{\psi_{k_j k_t}}{2}\right) = ctg\left(\frac{k_j}{2}\right) - ctg\left(\frac{k_t}{2}\right); \quad k_j = \frac{2\pi p + \sum_1^r \psi_{k_j k_t}}{N}.$$

$$E = \sum_{i=1}^{r} \varepsilon_{k_i}; \quad E_{ground} = \sum_{i=1}^{N/2} \varepsilon_{k_i} = 2|J|\left(N/4 - N\ln 2\right).$$

EXACT RESULTS FOR HEISENBERG EXCHANGE MODEL

Spin-1/2 chain with anisotropic exchange

$$\mathbf{H} = -g\mu_B H \sum_{n=1}^{N} S_n^z - \sum_{n=1}^{N} \left(J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z \right)$$

Ising model in longitudinal field (1925) $\left(J_x = J_y = 0,\ J_z \neq 0 \right)$

XXX-model: Bethe Ansatz (1931), Hulthén (1938) $\left(J_x = J_y = J_z = J \right)$

Ising model in transverse field $\left(J_x \neq 0,\ J_y = J_z = 0 \right)$

XY-model: Lieb, Shultz, Mattis (1961) $\left(J_x \neq J_y,\ J_z = 0 \right)$

XXZ-model: Orbach (1958), Yang, Yang (1966) $\left(J_x = J_y \neq J_z \right)$

XYZ-model: Baxter (1972), Takhtajan, Faddeev (1981) $\left(J_x \neq J_y \neq J_z \right)$

Spin-1/2 XY chain. Ideal gas of spinless fermions.

$$\mathbf{H} = -2\mu_B H \sum_{n=1}^{N} S_n^z - \sum_{n=1}^{N} \left(J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y \right)$$

Jordan-Wigner Transformation

$$S_n^{\pm} = S_n^x \pm i S_n^y; \quad S_n^+ = \prod_{l=1}^{n-1} \left(1 - 2a_l^{\dagger} a_l\right) a_l; \quad S_n^- = a_l^{\dagger} \prod_{l=1}^{n-1} \left(1 - 2a_l^{\dagger} a_l\right);$$

$$S_n^z = 1/2 - a_n^{\dagger} a_n; \quad \left[a_m, a_n\right]_+ = 0, \left[a_m^{\dagger}, a_n^{\dagger}\right]_+ = 0, \left[a_m^{\dagger}, a_n\right]_+ = \delta_{m,n}.$$

Fourier transformation

$$a_n = \frac{1}{\sqrt{N}} \sum \exp(ikn) a_k.$$

$$\mathbf{H} = -N\mu H + \sum \left[\left(2\mu H - \frac{J_x + J_y}{2} \cos k \right) a_k^{\dagger} a_k + i \frac{J_x - J_y}{4} \sin k \left(a_k a_{-k} - a_{-k}^{\dagger} a_k^{\dagger} \right) \right]$$

Spin-1/2 XY chain. Ideal gas of spinless fermions.

After Bogholubov u-v transformation

$$\begin{cases} a_k = U_k b_k + V_k^* b_{-k}^\dagger; & U_{-k} = U_k; V_{-k} = -V_k; \\ a_{-k} = U_k b_{-k} - V_k^* b_k^\dagger; & |U_k|^2 + |V_k|^2 = 1. \end{cases}$$

we get the Hamiltonian

$$\mathbf{H} = \sum_k \varepsilon_k \left(b_k^\dagger b_k - 1/2 \right), \quad \varepsilon_k = \sqrt{\left(2\mu_B H - \frac{(J_1 + J_2)}{2} \cos k \right)^2 + \frac{(J_1 - J_2)^2}{4} \sin^2 k};$$

$$Z = Tr = \exp\left[-\left(\frac{\mathbf{H}}{T} \right) \right]; \quad F = -T \ln Z; \quad M = -\frac{\partial F}{\partial H} = \frac{1}{2} \sum_k \frac{\partial \varepsilon_k}{\partial H} \tanh\left(\frac{\varepsilon_k}{T} \right);$$

$$\chi = \frac{\partial M}{\partial H}; \quad E = -T^2 \frac{\partial}{\partial T}\left(\frac{F}{T} \right); \quad C = \frac{\partial E}{\partial T}.$$

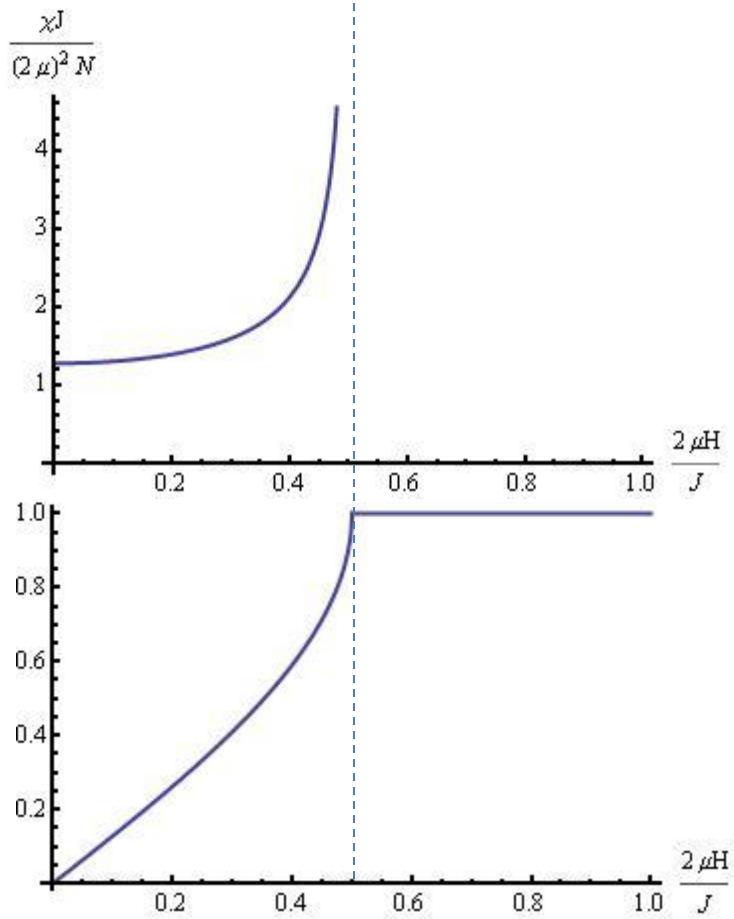
Spin-1/2 XX chain. Quantum Phase Transition at T = 0.

$$J_1 = J_2 = J, \quad \varepsilon_k = |2\mu H - J\cos k|;$$

$$M(H)/N = \begin{cases} \mu\left(1 - \dfrac{2}{\pi}\arccos\dfrac{2\mu H}{J}\right), & 2\mu H \le J; \\ \mu, & 2\mu H > J. \end{cases}$$

$$\chi(H)/N = \begin{cases} \dfrac{(2\mu)^2}{\pi J}\arccos\dfrac{2\mu H}{J}, & 2\mu H \le J; \\ 0, & 2\mu H > J. \end{cases}$$

Spin-1/2 XX chain. Quantum Phase Transition at T = 0.



❑Four main models for strong magnetism

- Lenz-Ising model (1925)

$$\mathbf{H} = -\sum_{i} h_i \sigma_i - \sum_{i \neq j} J_{i,j} \sigma_i \sigma_j , \quad (\sigma_i = \pm 1)$$

Lenz-Ising formulation of the problem of ferromagnetism: spins are disposed at regular intervals along the length of a *one-dimensional* chain. Each spin was allowed to take on the values ±1.

$$\mathbf{H} = -\sum_{n=1}^{N} h_n \sigma_n - \sum_{n=1}^{N} J_n \sigma_n \sigma_{n+1} , \quad (\sigma_n = \pm 1)$$

Ernst Ising solved the 1D Ising model in 1925 and found that there is no phase transition in the 1D model. With wrong arguments, he also concluded that 2D Ising models had no phase transitions either.

1.E.ISING, *Beitrag zur Theorie des Ferro- und Paramagnetismus Dissertation*, Mathematisch-Naturwissenschaftliche Fakultät der Hamburgischen Universität Hamburg, 1924 (unpublished)
See: http://www.hs-augsburg.de/~harsch/germanica/Chronologie/20Jh/Ising/isi_intr.html
2.E.ISING, *Beitrag zur Theorie des Ferromagnetismus*, Zeitschrift fűr Physik 31 (1925) 253-258.

❑Four main models for strong magnetism

Ising model in two dimensions.

Second order phase transition in ferromagnetic case.

A milestone in the development of the modern Statistical Mechanics is the ***exact solution*** of the 2D Ising model in a square lattice by Lars Onsager in 1944 in zero magnetic field.

$$M = \left\{ 1 - \left[\frac{\sinh\left(\ln(1+\sqrt{2})\right) T_c}{T} \right]^{-1} \right\}^{\frac{1}{8}}, \quad T_c = \frac{2J}{\left(\ln(1+\sqrt{2})\right)}$$

Interestingly, Onsager wrote on a blackboard at a conference in 1949 the critical exponent *s* =1/8. How he got the number remains mysterious since he did not present any follow-up publication on that. The first published calculation of *s*=1/8 was due to C. N. Yang in 1952.

L.ONSAGER. *Crystal statistics. I. A two-dimensional model with an order-disorder transition*. Phys.Rev. – 1944. - V.65, NN 3-4. - P. 117-149 .

C.N.YANG *The spontaneous magnetization of a two-dimensional Ising model*. Phys.Rev. – 1952. - V. 85, NN 5. - P. 808-816 .

❑Four main models for strong magnetism

One-band Hubbard model

The Hubbard model, named after John Hubbard, is the simplest model of interacting particles in a lattice, with only two terms in the Hamiltonian: a kinetic term allowing for tunneling ("hopping") of particles between sites of the lattice and a potential term consisting of an on-site interaction.

$$\mathbf{H} = \sum_{<i,j>,\sigma} t_{ij} \left(a^{\dagger}_{m,\sigma} a_{m+1,\sigma} + h.c. \right) + U \sum_{i} a^{\dagger}_{i,\sigma} a_{i,\sigma} a^{\dagger}_{i,-\sigma} a_{i,-\sigma}$$

Was proposed by Gutzwiller (1963), Hubbard (1963)
Exact solution for 1D chain with Hamiltoinan

$$\mathbf{H} = t \sum_{m=1,\sigma}^{N} \left(a^{\dagger}_{m,\sigma} a_{m+1,\sigma} + h.c. \right) + U \sum_{m=1}^{N} a^{\dagger}_{m,\sigma} a_{m,\sigma} a^{\dagger}_{m,-\sigma} a_{m,-\sigma}$$

Lieb , Wu (1968), Ovchinnikov (1970)

❑Four main models for strong magnetism

s-d model

$$\mathbf{H} = \sum_{<i,j>,\sigma} t_{ij}\left(a^{\dagger}_{m,\sigma} a_{m+1,\sigma} + h.c.\right) + J\sum_{i} \mathbf{S}_i \mathbf{s}_i$$

Was proposed by Vonsovskiy in 1946.

The first term represents conduction electrons (s-electrons in the original model) hopping from site *i* to site *j* The second term contains localized spin operators $\mathbf{S}_i$ (corresponding to the d-electrons in the original model) and spin operators $\boldsymbol{s}_{\mathrm{i}}$ of a conduction electron that interacts with the localized electron $\mathbf{S}_{\mathrm{i}}$, *J* (ferromagnetic interaction).

The s-d model is only exactly solvable in one dimension using the Bethe ansatz , as Andrei and Wiegmann showed independently (1980). The s-d model consists of a lattice of paramagnetic impurities that interact with conduction electrons.