Quantum mechanics. Department of physics. 7th semester.

Lesson № 8-9. ***Refresher course*** ***of previous semester: main equations and concepts of quantum mechanics.***

1. De Broglie wavelength (matter waves).
2. Wave function. The physical meaning of the wave function.
3. Definition of Hermitian operator. Eigenvalues and eigenfunctions equations. Properties of eigenvalues and eigenfunctions of Hermitian operator.
4. Average value of the operator of a physical quantity.
5. Schrodinger’s wave equation.
6. Schrodinger’s stationary equation. Hamilton’s operator.
7. Operators of coordinates and impulse.
8. Hamiltonian of the harmonic oscillator.
9. Uncertainty relation between coordinate and impulse.
10. Spin concept. Pauli exclusion principle.
11. Qualitative explain character of the particle movement in a field with potential energy $U(x)$ of the set type.

New topic**: *Angular momentum operator. Commutation relations. Eigenfunctions and eigenvalues of $\hat{l}_z$ и $\hat{\vec{l}}^{\,2}$. Raising and lowering (ladder) operators $\hat{l}_\pm$.***

1. Angular momentum operator.

1.1. Definition of the angular momentum operator $\hat{\vec{L}}$ in Cartesian coordinates:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix}; \quad \hat{\vec{r}} = \vec{r}, \quad \hat{\vec{p}} = -i\hbar\nabla = -i\hbar\frac{\partial}{\partial\vec{r}},$$

In tensor notation:

$$\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k,$$

ε_{ijk} – Levi-Civita symbol.

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$
$$\hat{\vec{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2.$$

Task 1. Prove commutation relation:

$$\left[\hat{L}_i, \hat{x}_j\right] = i\hbar\varepsilon_{ijk}\hat{x}_k; \quad \left[\hat{L}_i, \hat{p}_j\right] = i\hbar\varepsilon_{ijk}\hat{p}_k; \quad \left[\hat{L}_i, \hat{L}_j\right] = i\hbar\varepsilon_{ijk}\hat{L}_k; \quad \left[\hat{\vec{L}}^2, \hat{L}_i\right] = 0.$$

Task 2. Calculate $\left[\hat{L}_i, \hat{\vec{r}}^{\,2}\right]$ (From НКК № 3.4(a))

1.2. Dimensionless angular momentum operator $\hat{\vec{l}}$:

$$\hat{\vec{l}} = \frac{1}{\hbar}\hat{\vec{L}}; \quad \left[\hat{l}_i, \hat{l}_j\right] = i\varepsilon_{ijk}\hat{l}_{jk}; \quad \left[\hat{\vec{l}}^{\,2}, \hat{l}_i\right] = 0.$$

1.3. Operators $\hat{l}_z$ и $\hat{\vec{l}}^{\,2}$ in spherical coordinates:

$$\hat{l}_z = -i\frac{\partial}{\partial\varphi}; \quad \hat{\vec{l}}^{\,2} = -\Delta_{\theta\varphi} = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right].$$

1.4. $\hat{l}_z$ and $\hat{\vec{l}}^{\,2}$ eigenfunctions and eigenvalues:

$$\hat{\vec{l}}^{\,2} Y_{lm}(\theta,\varphi) = l(l+1)Y_{lm}(\theta,\varphi), \quad l = 0,1,2,3,...;$$
$$\hat{l}_z Y_{lm}(\theta,\varphi) = mY_{lm}(\theta,\varphi), \quad m = 0,\pm 1,\pm 2,...,\pm l.$$

l – orbital (azimuthal) quantum number, m – magnetic quantum number.

Spherical function: $Y_{lm}(\theta,\varphi) = C_{lm}P_l^m(\theta)e^{im\varphi}$;

$$P_l^m\left(\cos\theta\right) = \sin^m\theta\frac{d^m}{\left(d\cos\theta\right)^m}P_l\left(\cos\theta\right); \quad P_l\left(\cos\theta\right) = \frac{1}{2^l l!}\frac{d^l}{\left(d\cos\theta\right)^l}\left(\cos^2\theta - 1\right)^l,$$

где $P_l^m(\theta)$ – associated Legendre polynomials, $P_l\left(\cos\theta\right)$ – Legendre polynomials, C_{lm} – normalization constant.

1.4. Raising and lowering (ladder) operators $\hat{l}_\pm$:

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y.$$

Task 3. Prove commutation relation

$$\left[\hat{l}_z,\hat{l}_\pm\right]=\pm\hat{l}_\pm; \quad \left[\hat{l}_+,\hat{l}_-\right]=2\hat{l}_z.$$

Task 4. Prove, that $\hat{l}_\pm\psi_m$, where ψ_m – eigenfunctions of the operator $\hat{l}_z$ z-projections of the angular momentum $\left(\hat{l}_z\psi_m = m\psi_m, \quad m=0,\pm1,\pm2,...\right)$, are also eigenfunctions of the operator $\hat{l}_z$ with eigenvalues $m+1$ and $m-1$ for $\hat{l}_+$ и $\hat{l}_-$ (НКК №3.11)

Task 5. Find eigenvalues of the operator $\hat{\vec{l}}^{\,2}$. Use commutation relations from **task 3.**

Task 6. In state ψ_m with certain projection of the angular momentum on the axis z calculate average values $\overline{l_x}, \overline{l_y}, \overline{l_z}, \overline{l_x l_y}, \overline{l_y l_x}, \overline{l_x^2}, \overline{l_y^2}, \overline{l_z^2}$. (НКК №3.12)

Hometask НКК № 3.4(a), НКК № 3.12 (finish), 3.13, 3.14.

НКК- Halitskii E.M., Karnakov B.M., Kohan V.I. Problems in Quantum Physics, 1981

Hr. - Hrechko, Suhakov, Tomasevich, Fedorchenko Collection of theoretical physics problems, 1984