Quantum mechanics. Department of physics, 6th semester.

Lesson №4. ***Mathematical tools of quantum mechanics: calculating average values of operators. Elements of representation theory. Discrete and continuous representations.***

1. Check home task.

Task 1. Find a Hermitian conjugated operator to the operator $e^{i\varphi\hat{\sigma}_j}$.

Tasks 2-3. Find eigenfunctions and eigenvalues of matrices $\hat{\sigma}_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$; $\hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$.

2. Calculating average values of operators.

$$Def.: \quad \overline{A} = \left(\psi, \hat{A}\psi\right), \quad \overline{A^2} = \left(\psi, \hat{A}^2\psi\right),$$

$$\overline{\Delta A^2} = \overline{\left(\hat{A} - \overline{A}\right)^2} = \overline{A^2} - \left(\overline{A}\right)^2, \quad \delta A = \sqrt{\overline{\Delta A^2}}.$$

Task 4. In described with wave function state

$$\psi(x) = C\exp\left[\frac{ip_0 x}{\hbar} - \frac{(x - x_0)^2}{2a^2}\right],$$

where p_0, x_0, a – real-valued parameters, find distribution function in the coordinates of the particle. Define $\overline{x}, \overline{x^2}, \overline{p}, \overline{p^2}, \overline{\Delta x^2}, \overline{\Delta p^2}, \delta x, \delta p, \delta x \cdot \delta p.$ (НКК № 1.19)

As reference: $\int\limits_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}, \quad \alpha > 0.$

3. Elements of representation theory.

3.1 Discrete representation.

$$\hat{L}^\dagger = \hat{L}, \quad \hat{L}\psi_n = \lambda_n \psi_n; \quad \left(\psi_m, \psi_n\right) = \delta_{mn}.$$

$$\begin{cases} \psi(x) = \sum\limits_n C_n \psi_n; \\ C_n = \left(\psi_n, \psi\right) = \int\limits_{-\infty}^{+\infty} \psi_n^*(x)\psi(x)dx; \end{cases} \qquad \begin{cases} \sum\limits_n \psi_n^*(x')\psi_n(x) = \delta(x - x'); \\ \int\limits_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x)dx = \delta_{mn}. \end{cases}$$

$\left\{C_n\right\}$ – function $\psi(x)$ in discrete *L*-representation,

$A_{mn} = \left(\psi_m, \hat{A}\psi_n\right) = \int\limits_{-\infty}^{+\infty} \psi_m^*(x)\hat{A}\psi_n(x)dx$ is an operator $\hat{A}$ matrix in discrete *L*-representation.

$L_{mn} = \lambda_n \delta_{mn}$ - operator $\hat{L}$ matrix in own representation.

$$\hat{A}\psi(x) = \tilde{\psi}(x) \quad \rightarrow \quad \sum_n A_{mn}C_n = \tilde{C}_m, \quad C_n = \left(\psi_n, \psi\right), \tilde{C}_m = \left(\psi_n, \tilde{\psi}\right)$$

<u>**Task 5.**</u> Rewrite Pauli's matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ in representation of eigenfunctions of the matrix $\hat{\sigma}_x$, $\hat{\sigma}_y$.

3.2. Continuous representation.

$$\hat{L}^\dagger = \hat{L}, \quad \hat{L}\psi_\lambda = \lambda\psi_\lambda; \quad \left(\psi_\lambda, \psi_{\lambda'}\right) = \delta(\lambda - \lambda').$$

$$\begin{cases} \psi(x) = \int\limits_{-\infty}^{+\infty} C(\lambda)\psi_\lambda(x)d\lambda \\ C(\lambda) = \left(\psi_\pi, \psi\right) = \int\limits_{-\infty}^{+\infty} \psi_\lambda^*(x)\psi(x)dx; \end{cases}$$

$$\begin{cases} \int\limits_{-\infty}^{+\infty} \psi_\lambda^*(x')\psi_\lambda(x)d\lambda = \delta(x - x'); \\ \int\limits_{-\infty}^{+\infty} \psi_{\lambda'}^*(x)\psi_\lambda(x)dx = \delta(\lambda - \lambda'); \end{cases}$$

$C(\lambda)$ – function $\psi(x)$ in discrete L-representation,

$A(\lambda, \lambda') = \left(\psi_\lambda, \hat{A}\psi_{\lambda'}\right) = \int\limits_{-\infty}^{+\infty} \psi_\lambda^*(x)\hat{A}\psi_{\lambda'}(x)dx$ is a $\hat{A}$ operator kernel in continuous L-representation

$L(\lambda, \lambda') = \lambda\delta(\lambda - \lambda')$ is a $\hat{L}$ kernel in own representation.

$$\hat{A}\psi(x) = \tilde{\psi}(x) \quad \rightarrow \quad \int\limits_{-\infty}^{+\infty} A(\lambda, \lambda')C(\lambda')d\lambda' = \tilde{C}(\lambda), \quad C(\lambda) = \left(\psi_\lambda, \psi\right), \tilde{C}(\lambda') = \left(\psi_{\lambda'}, \tilde{\psi}\right).$$

Operator $\hat{L}$ in its own continuous representation is the multiplication by λ $\hat{L}C(\lambda) = \lambda C(\lambda)$.

4. Dirac delta-function is the kernel of the unity operator.

Properties of the Dirac delta-function

$$Def: \quad \psi(a) = \int_{-\infty}^{+\infty} \psi(x)\delta(x-a)dx;$$

$$\delta(-x) = \delta(x); \quad \int_{-\infty}^{+\infty} \delta(x)dx = 1; \quad \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x);$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dx = \delta(k); \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk = \delta(x).$$

<u>Tasks 6-7.</u> Find position $\hat{x}$ and momentum $\hat{p}$ operators in *p*-representation. As reference: momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$, normalized on δ–function Eigenfunction has the form

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}.$$

5.**<u>Quiz</u>** (~ 20 minutes). Test contains two tasks: 1st task is 10 points, 2nd task - 10 points, to sum up maximum **<u>20 points</u>**.

<u>Home task</u>: НКК №№ 1.19 (to finish), 1.22-1.25, 1.30, 1.42, 1.44, 1.45, 1.46*, 1.47*, 1.48*,1.54-1.59, 1.67*, Hr. № 32.

EK – Elyutin P.V., Krivchenko V.D. Quantum mechanics 1976
НКК- Halitskii E.M., Karnakov B.M., Kohan V.I. Problems in Quantum Physics, 1981
Hr. - Hrechko, Suhakov, Tomasevich, Fedorchenko Collection of theoretical physics problems, 1984

*) – tasks for students of group Ф037