Quantum mechanics. Department of physics. 7th semester.

Lesson №13. ***Perturbation theory (PT): time-dependent PT.***

1. Preparation for Midterm. Test consists of 3 tasks:

1.Theoretical question from «Quantum minimum» (**3 points**); 2. Calculating commutator task (**3 points**); 3. One-dimensional motion task (**4 points**). Highest mark – **10 points.**

2. Perturbation theory

$$\hat{H} = \hat{H}_0 + \hat{V}; \quad \hat{H}_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)};$$

$\hat{H}_0$ – basic Hamiltonian, which exact solution of the stationary Schrödinger equation is known for, $\hat{V}$ – perturbation operator.

$\dfrac{\partial \hat{V}}{\partial t} = 0$ – time-independent PT (see lesson № 12), $\dfrac{\partial \hat{V}}{\partial t} \neq 0$ – time-dependent PT.

3. Time-dependent PT.

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \left(\hat{H}_0 + \hat{V}(t) \right) \Psi(q,t);$$

$$\Psi(q,t) = \sum_k a_k(t) \Psi_k^{(0)}(q,t); \quad \Psi_k^{(0)}(q,t) = \psi_k^{(0)}(q) e^{-iE_k^{(0)} t/\hbar},$$

$$i\hbar \frac{da_m}{\partial t} = \sum_k V_{mk}(t) a_k, \qquad .$$

$$V_{mk}(t) = \left(\Psi_m^{(0)}, \hat{V} \Psi_k^{(0)} \right) = V_{mk} e^{i\omega_{mk} t}, \quad \omega_{mk} = \frac{E_m^{(0)} - E_k^{(0)}}{\hbar}$$

$$a_{kn} \approx \delta_{k,n} + a_{kn}^{(1)}(t), \quad a_{kn}^{(1)} = -\frac{i}{\hbar} \int V_{kn} e^{i\omega_{kn} t} dt.$$

3.1. Probability of a transition under the influence of perturbations acting for a finite time

$$w_{k \leftarrow n} = | a_{kn}^{(1)} |^2 = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} V_{kn} e^{i\omega_{kn} t} dt \right|^2 .$$

Task 1. A particle, being at $t \to -\infty$ in the ground state in an infinitely deep well with a width $a\,(0 < x < a)$, is imposed weak homogeneous field, changing in time according to the law $V(x,t) = -xF_0 e^{-t^2/\tau^2}$. In first order of PT calculate probability transitions of the particle in the limit $t \to \infty$ into different exited states. (НКК № 8.23 (a))

4. Discussion of individual home tasks. Some information about special functions, which can be useful while doing individual tasks.

4.1. Hypergeometric functions.

Hypergeometric series (generalization of the concept of geometric progression)

$$F(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\cdot\beta}{\gamma\cdot 1} z + \frac{\alpha\cdot(\alpha+1)\cdot\beta\cdot(\beta+1)}{\gamma\cdot(\gamma+1)\cdot 1\cdot 2} z^2 + \frac{\alpha\cdot(\alpha+1)\cdot(\alpha+2)\cdot\beta\cdot(\beta+1)\cdot(\beta+2)}{\gamma\cdot(\gamma+1)\cdot(\gamma+2)\cdot 1\cdot 2\cdot 3} z^3 + \ldots;$$

The series converges when $|z| \le 1$, is symmetrical at α, β. The series terminates, if α or β is integer negative number. This is one of the solutions to the hypergeometric equation

$$z(1-z)\frac{d^2U}{dz^2} + \left[\gamma - (\alpha+\beta+1)z\right]\frac{dU}{dz} - \alpha\beta U = 0.$$

Two linear independent solutions in case γ is not integer number,

$$U_1(z) = F(\alpha,\beta,\gamma,z);$$
$$U_2(z) = z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z).$$

Through hypergeometric function one expresses complete elliptic integrals, integrals of cylindrical functions, the Legendre polynomials, Legendre functions, spherical functions and adjoint Legendre polynomials, Chebyshev polynomials, Jacobi polynomials. For example,

$$P_n(z) = \frac{(2n-1)!!}{n!} z^n F\left(-\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2}-n, \frac{1}{z^2}\right);$$
$$P_n(\cos\varphi) = F\left(n+1, -n, 1, \sin^2\varphi\right).$$

4.2. Confluent hypergeometric function.

Confluent hypergeometric series

$$F(\alpha,\gamma,z)=1+\frac{\alpha}{\gamma}\cdot\frac{z}{1!}+\frac{\alpha\cdot(\alpha+1)}{\gamma\cdot(\gamma+1)}\cdot\frac{z^2}{2!}+\frac{\alpha\cdot(\alpha+1)\cdot(\alpha+2)}{\gamma\cdot(\gamma+1)\cdot(\gamma+2)}\cdot\frac{z^3}{3!}+\ldots;$$

The series converges when $|z|\leq 1$. The series terminates, if α is integer negative number. It's one of the confluent hypergeometric equation solutions

$$z\frac{d^2U}{dz^2}+(\gamma-z)\frac{dU}{dz}-\alpha U=0.$$

If γ is not an integer number, then two linear independent solutions of the confluent hypergeometric equation is like:

$$U_1(z)=F(\alpha,\gamma,z);$$
$$U_2(z)=z^{1-\gamma}F(\alpha-\gamma+1,2-\gamma,z).$$

Through the confluent hypergeometric function one expresses integrals of probability, integrals of cylindrical functions, Hermite polynomials, polynomials of Laguerre.

More detailed about properties of hypergeometric function and confluent hypergeometric function one can read in: I. S. Gradshteyn, I. M. Ryzhik "Table of integrals, sums, series and products", A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, "Integrals and series. Special functions. Vol. 2", M. Abramowitz and I. Stigan "Handbook of special functions", as well as in mathematical additions in the textbook "Quantum mechanics" L. D. Landau and I. M. Lifshitz.

Hometasks. НКК 8.23(b, v)–8.26.

LL – Landau LD, Lifshits EM, Quantum Mechanics

НКК- Halitskii E.M., Karnakov B.M., Kohan V.I. Problems in Quantum Physics, 1981